

A SUB-RIEMANNIAN CURVATURE-DIMENSION INEQUALITY, VOLUME DOUBLING PROPERTY AND THE POINCARÉ INEQUALITY

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ABSTRACT. Let \mathbb{M} be a smooth connected manifold endowed with a smooth measure μ and a smooth locally subelliptic diffusion operator L satisfying $L1 = 0$, and which is symmetric with respect to μ . We show that if L satisfies, with a non negative curvature parameter, the generalized curvature inequality introduced by the first and third named authors in [BG1], then the following properties hold:

- The volume doubling property;
- The Poincaré inequality;
- The parabolic Harnack inequality.

The key ingredient is the study of dimensional reverse log-Sobolev inequalities for the heat semigroup and corresponding non-linear reverse Harnack type inequalities. Our results apply in particular to all Sasakian manifolds whose horizontal Webster-Tanaka-Ricci curvature is non negative, all Carnot groups with step two, and to wide subclasses of principal bundles over Riemannian manifolds whose Ricci curvature is non negative.

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1. INTRODUCTION

A fundamental property of a measure metric space (X, d, μ) is the so-called *doubling condition* stating that for every $x \in X$ and every $r > 0$ one has

$$(1.1) \quad \mu(B(x, 2r)) \leq C_d \mu(B(x, r)),$$

for some constant $C_d > 0$, where $B(x, r) = \{x \in X \mid d(y, x) < r\}$. As it is well-known, such property is central for the validity of covering theorems of Vitali-Wiener type, maximal function estimates, and it represents one of the central ingredients in the development of analysis and geometry on metric measure spaces, see for instance [Fe], [FS], [CW], [HK], [He], [Ha], [AT].

First author supported in part by NSF Grant DMS 0907326.

Third author supported in part by NSF Grant DMS-1001317.

Another fundamental property is the *Poincaré inequality* which claims the existence of constants $C_p > 0$ and $a \geq 1$ such that for every Lipschitz function f on $B(x, ar)$ one has

$$(1.2) \quad \int_{B(x,r)} |f - f_B|^2 d\mu \leq C_p r \int_{B(x,ar)} g^2 d\mu,$$

where we have let $f_B = \mu(B)^{-1} \int_B f d\mu$, with $B = B(x, r)$. In the right-hand side of (1.2) the function g indicates an *upper gradient* for f (see [Che] and [HeK] for a discussion of upper gradients).

One basic instance of a measure metric space supporting (1.1) and (1.2) is a complete n -dimensional Riemannian manifold \mathbb{M} with nonnegative Ricci tensor. In such case (1.1) follows with $C_d = 2^n$ from the Bishop-Gromov comparison theorem (see e.g. Theorem 3.10 in [Cha]), whereas (1.2) was proved by Buser in [Bu], with $a = 1$ and $g = |\nabla f|$.

Beyond the classical Riemannian case two situations of considerable analytic and geometric interest are CR and sub-Riemannian manifolds. For these classes global inequalities such as (1.1) and (1.2) are mostly *terra incognita*. The purpose of the present paper is taking a first step in filling this gap.

To introduce our results we consider measure metric spaces (\mathbb{M}, d, μ) , where \mathbb{M} is C^∞ manifold endowed with a C^∞ measure μ , and d is a metric canonically associated with a C^∞ second-order diffusion operator L on \mathbb{M} with real coefficients. We assume that L is locally subelliptic on \mathbb{M} , and that moreover:

- (i) $L1 = 0$;
- (ii) $\int_{\mathbb{M}} f Lg d\mu = \int_{\mathbb{M}} g Lf d\mu$;
- (iii) $\int_{\mathbb{M}} f Lf d\mu \leq 0$,

for every $f, g \in C_0^\infty(\mathbb{M})$. The distance d is constructed as follows:

$$(1.3) \quad d(x, y) = \sup \{ |f(x) - f(y)| \mid f \in C^\infty(\mathbb{M}), \|\Gamma(f)\|_\infty \leq 1 \}, \quad x, y \in \mathbb{M},$$

where for a function g on \mathbb{M} we have let $\|g\|_\infty = \text{ess sup}_{\mathbb{M}} |g|$.

The quadratic functional $\Gamma(f) = \Gamma(f, f)$, where

$$(1.4) \quad \Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf), \quad f, g \in C^\infty(\mathbb{M}),$$

is known as *le carré du champ*. Notice that $\Gamma(f) \geq 0$ and that $\Gamma(1) = 0$.

Throughout this paper we assume that the metric space (M, d) be complete. We also suppose that \mathbb{M} is equipped with a symmetric, first-order differential bilinear form $\Gamma^Z : C^\infty(\mathbb{M}) \times C^\infty(\mathbb{M}) \rightarrow \mathbb{R}$, satisfying

$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h).$$

We assume that $\Gamma^Z(f) = \Gamma^Z(f, f) \geq 0$ (one should notice that $\Gamma^Z(1) = 0$).

Given the first-order bilinear forms Γ and Γ^Z on \mathbb{M} , we now introduce the following second-order differential forms:

$$(1.5) \quad \Gamma_2(f, g) = \frac{1}{2} [L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)],$$

$$(1.6) \quad \Gamma_2^Z(f, g) = \frac{1}{2} [L\Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)].$$

Observe that if $\Gamma^Z \equiv 0$, then $\Gamma_2^Z \equiv 0$ as well. As for Γ and Γ^Z , we will use the notations $\Gamma_2(f) = \Gamma_2(f, f)$, $\Gamma_2^Z(f) = \Gamma_2^Z(f, f)$.

The next definition represents the central notion of this paper.

Definition 1.1. *We say that \mathbb{M} satisfies the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ if there exist constants $\rho_1 \in \mathbb{R}$, $\rho_2 > 0$, $\kappa \geq 0$, and $d \geq 2$ such that the inequality*

$$(1.7) \quad \Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \frac{1}{d} (Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu} \right) \Gamma(f) + \rho_2 \Gamma^Z(f)$$

hold for every $f \in C^\infty(\mathbb{M})$ and every $\nu > 0$.

Let us observe right-away that if $\rho'_1 \geq \rho_1$, then $CD(\rho'_1, \rho_2, \kappa, d) \implies CD(\rho_1, \rho_2, \kappa, d)$. To provide the reader with some perspective on Definition 1.1 we remark that it constitutes a generalization of the so-called *curvature-dimension inequality* $CD(\rho_1, n)$ from Riemannian geometry, see [BE], [Ba], [St1], [St2], [LV]. We recall that the latter is said to hold on a n -dimensional Riemannian manifold \mathbb{M} with Laplacian Δ if there exists $\rho_1 \in \mathbb{R}$ such that for every $f \in C^\infty(\mathbb{M})$ one has

$$(1.8) \quad \Gamma_2(f) \geq \frac{1}{n} (\Delta f)^2 + \rho_1 |\nabla f|^2,$$

where

$$\Gamma_2(f) = \frac{1}{2} (\Delta |\nabla f|^2 - 2 \langle \nabla f, \nabla(\Delta f) \rangle).$$

To see that (1.7) contains (1.8) it is enough to take $L = \Delta$, $\Gamma^Z = 0$, $\kappa = 0$, and $d = n$, and notice that (1.4) gives $\Gamma(f) = |\nabla f|^2$ (also note that in this context the distance (1.3) is simply the Riemannian distance on \mathbb{M}). It is worth emphasizing at this moment that, remarkably, on a complete Riemannian manifold the inequality (1.8) is equivalent to the lower bound $\text{Ric} \geq \rho_1$. One direction easily follows from the Bochner identity and the Cauchy-Schwarz inequality. The opposite direction is more complicated, but it follows by combining Theorem 1.3 in [RS] with Proposition 3.3 in [Ba].

The essential new aspect of the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ with respect to the Riemannian inequality $CD(\rho_1, n)$ in (1.8) is the presence of the a priori non-intrinsic forms Γ^Z and Γ_2^Z . In the non-Riemannian framework of this paper the form Γ plays the role of the square of the length of a gradient along the (horizontal) directions canonically associated with the operator L , whereas the form Γ^Z should be thought of as the square of the length of a gradient in the missing (vertical) directions.

In Definition 1.1 the parameter ρ_1 plays a special role. For the results in this paper such parameter represents the lower bound on a sub-Riemannian generalization of the Ricci tensor. Thus, $\rho_1 \geq 0$ is, in our framework, the counterpart of the Riemannian $\text{Ric} \geq 0$. For this reason, when in this paper we say that \mathbb{M} satisfies the curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \geq 0$, we will routinely avoid repeating at each occurrence the sentence “for some $\rho_2 > 0$, $\kappa \geq 0$ and $d \geq 2$ ”.

Before stating our main result we need to introduce two further technical assumptions on the forms Γ and Γ^Z :

(H.1) There exists an increasing sequence $h_k \in C_0^\infty(\mathbb{M})$ such that $h_k \nearrow 1$ on \mathbb{M} , and

$$\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

(H.2) For any $f \in C^\infty(\mathbb{M})$ one has

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

The hypothesis (H.1) and (H.2) will be in force throughout the paper. Let us notice explicitly that when \mathbb{M} is a complete Riemannian manifold with $L = \Delta$, then (H.1) and (H.2) are fulfilled. In fact, (H.2) is trivially satisfied since we can take $\Gamma^Z \equiv 0$, whereas (H.1) follows from (and it is in fact equivalent to) the completeness of (\mathbb{M}, d) . Actually, more generally, in the geometric examples encompassed by the framework of this paper (for a detailed discussion of these examples the reader can consult [BG1]), (H.1) is equivalent to assuming that (\mathbb{M}, d) be a complete metric

space (the reason is that in those examples $\Gamma + \Gamma^Z$ is the *carré du champ* of the Laplace-Beltrami of a Riemannian structure whose completeness is equivalent to the completeness of (\mathbb{M}, d)).

In this paper we also assume that given any two points $x, y \in \mathbb{M}$, there exist a subunit curve joining them. Under such assumption, the metric space (\mathbb{M}, d) is a length-space in the sense of Gromov. For these notions we refer the reader to Section 2.1 below. We are now ready to state the main result in this paper.

Theorem 1.2. *Suppose that the generalized curvature-dimension inequality hold for some $\rho_1 \geq 0$. Then, there exist constants $C_d, C_p > 0$, depending only on $\rho_1, \rho_2, \kappa, d$, for which one has for every $x \in \mathbb{M}$ and every $r > 0$:*

$$(1.9) \quad \mu(B(x, 2r)) \leq C_d \mu(B(x, r));$$

$$(1.10) \quad \int_{B(x, r)} |f - f_B|^2 d\mu \leq C_p r^2 \int_{B(x, r)} \Gamma(f) d\mu,$$

for every $f \in C^1(\overline{B}(x, r))$.

We note explicitly that the possibility of having the same ball in both sides of (1.10) is due to the above mentioned fact that (\mathbb{M}, d) is a length-space. To put Theorem 1.2 in the proper perspective we note that, besides the already cited case of a complete Riemannian manifold having $\text{Ric} \geq 0$, the only genuinely sub-Riemannian manifolds in which (1.9) and (1.10) are presently known to simultaneously hold are stratified nilpotent Lie groups, aka Carnot groups. In such Lie groups the doubling condition (1.9) follows from a simple rescaling argument based on the non-isotropic group dilations, from the group left-translations and from the fact that the push-forward to the group of the Lebesgue measure on the Lie algebra is a bi-invariant Haar measure. Similar tools were also at the basis of Varopoulos' elementary proof of the Poincaré inequality (1.10) in [V] for more general Lie groups with polynomial growth.

It is worth mentioning at this point that, when L is a sum of square of vector fields like in Hörmander's work on hypoellipticity [H], then a local (both in $x \in X$ and $r > 0$) doubling condition was proved in the paper [NSW]. In this same framework, a local version of the Poincaré inequality was proved by D. Jerison in [J]. But no geometry is of course involved in these fundamental local results. The novelty of the present work is in the global character of the estimates (1.9) and (1.10).

We recall that one of the main motivations for the work [NSW] was understanding boundary value problems coming from several complex variables and CR geometry. In connection with CR manifolds we mention that in the recent paper [BG1] two of us have proved the following result.

Theorem 1.3. *Let (\mathbb{M}, θ) be a CR manifold with real dimension $2n + 1$ and vanishing Tanaka-Webster torsion, i.e., a Sasakian manifold. If for every $x \in \mathbb{M}$ the Tanaka-Webster Ricci tensor satisfies the bound*

$$\text{Ric}_x(v, v) \geq \rho_1 |v|^2,$$

for every horizontal vector $v \in \mathcal{H}_x$, then the curvature-dimension inequality $\text{CD}(\rho_1, \frac{n}{2}, 1, 2n)$ holds.

By combining Theorem 1.2 with Theorem 1.3 we obtain the following result.

Theorem 1.4. *Let \mathbb{M} be a Sasakian manifold of real dimension $2n + 1$. If for every $x \in \mathbb{M}$ the Tanaka-Webster Ricci tensor satisfies the bound $\text{Ric}_x \geq 0$, when restricted to the horizontal subbundle \mathcal{H}_x , then there exist constants $C_d, C_p > 0$, depending only on n , for which one has for every $x \in \mathbb{M}$ and every $r > 0$:*

$$(1.11) \quad \mu(B(x, 2r)) \leq C_d \mu(B(x, r));$$

$$(1.12) \quad \int_{B(x,r)} |f - f_B|^2 d\mu \leq C_p r^2 \int_{B(x,r)} |\nabla_H f|^2 d\mu.$$

In (1.12) we have denoted with $\nabla_H f$ the horizontal gradient of a function $f \in C^1(\overline{B}(x,r))$. Concerning Theorem 1.4 we mention that in their recent work [AL] Agrachev and Lee, with a completely different approach from us, have obtained (1.11) and (1.12) for three-dimensional Sasakian manifolds.

Once Theorem 1.2 is available, then from the work of Grygor'yan [Gri1] and Saloff-Coste [SC] (see also [FS], [KS1], [St1], [St2], [St3]) it is well-known that, in a very general Markov setting, the conjunction of (1.9) and (1.10) is equivalent to Gaussian lower bounds and uniform Harnack inequalities for the heat equation $L - \partial_t$. For the relevant statements we refer the reader to Theorems 4.1 and 4.4 below.

Another basic result which follows from Theorem 1.2 is a generalized Liouville type theorem, see Theorem 5.2 below, stating that, for any given $N \in \mathbb{N}$,

$$(1.13) \quad \dim \mathcal{H}_N(\mathbb{M}, L) < \infty,$$

where we have indicated with $\mathcal{H}_N(\mathbb{M}, L)$ the linear space of L -harmonic functions on \mathbb{M} with polynomial growth of order $\leq N$ with respect to the distance d .

In closing we mention that the framework of the present paper is analogous to that of the work [BG1], where two of us have used the generalized curvature-dimension inequality in Definition 1.1 to establish various global properties such as:

- (i) An a priori Li-Yau gradient estimate for solutions of the heat equation $L - \partial_t$ of the form $u(x, t) = P_t f(x)$, where P_t is the heat semigroup associated with L ;
- (ii) A scale invariant Harnack inequality for solutions of the heat equation of the form $u = P_t f$, with $f \geq 0$;
- (iii) A Liouville type theorem for solutions of $L f = 0$ on \mathbb{M} ;
- (iv) Off-diagonal upper bounds for the fundamental solution of $L - \partial_t$;
- (v) A Bonnet-Myers compactness theorem for the metric space (M, d) .

While we refer the reader to [BG1] for the detailed results, we mention here that at the time [BG1] was completed one major missing piece was precisely Theorem 1.2. In this sense the present paper should be seen as a completion of the program initiated in [BG1]. As for the ideas involved in the proof of Theorem 1.2 we mention that our approach is purely analytical and it is exclusively based on some new entropy functional inequalities for the heat semigroup. Our central result in the proof of Theorem 1.2 is a uniform Hölder estimate of the caloric measure associated with the diffusion operator L . Such estimate is contained in Theorem 3.7 below, and it states the existence of an absolute constant $A > 0$, depending only the parameters in the inequality $\text{CD}(\rho_1, \rho_2, \kappa, d)$, such that for every $x \in \mathbb{M}$, and $r > 0$,

$$(1.14) \quad P_{Ar^2}(\mathbf{1}_{B(x,r)})(x) \geq \frac{1}{2}.$$

Here, for a set $E \subset \mathbb{M}$, we have denoted by $\mathbf{1}_E$ its indicator function. Once the crucial estimate (1.14) is obtained, with the help of the Harnack inequality

$$(1.15) \quad P_s f(x) \leq P_t f(y) \left(\frac{t}{s} \right)^{\frac{D}{2}} \exp \left(\frac{D}{d} \frac{d(x, y)^2}{4(t-s)} \right), \quad s < t,$$

that was proved in [BG1] (for an explanation of the parameter D see (2.4) below), the proofs of (1.9), (1.10) become fairly standard, and they rely on a powerful circle of ideas that may be found in the literature.

The proof of (1.14) which represents the main novel contribution of the present work is rather technical. We mention that the main building block is a dimensional reverse logarithmic Sobolev inequality in Proposition 2.6 below. We stress here that, even in the Riemannian case, which is

of course encompassed by the present paper, such estimates are new and lead to some delicate reverse Harnack inequalities which constitute the key ingredients in the proof of (1.14). Still in connection with the Riemannian case, it is perhaps worth noting that, although as we have mentioned, in this setting the inequalities (1.1), (1.2) are of course well-known, nonetheless our approach provides a new perspective based on a systematic use of the heat semigroup. The more pde oriented reader might in fact find somewhat surprising that one can develop the whole local regularity starting from a global object such the heat semigroup. This in a sense reverses the way one normally proceeds, starting from local solutions.

Finally, we mention that in the recent paper [BG2] two of us have obtained a purely analytical proof of (1.14) for complete Riemannian manifolds with $\text{Ric} \geq 0$. The approach in that paper, which is based on a functional inequality much simpler than the one found in this paper, is completely different from that of Theorem 3.7 below and cannot be adapted to the non-Riemannian setting of the present paper.

2. REVERSE LOGARITHMIC SOBOLEV INEQUALITIES FOR THE HEAT SEMIGROUP

2.1. Framework. Hereafter in this paper, \mathbb{M} will be a C^∞ connected manifold endowed with a smooth measure μ and a second-order diffusion operator L on \mathbb{M} with real coefficients, locally subelliptic, satisfying $L1 = 0$ and

$$\int_{\mathbb{M}} f L g d\mu = \int_{\mathbb{M}} g L f d\mu, \quad \int_{\mathbb{M}} f L f d\mu \leq 0,$$

for every $f, g \in C_0^\infty(\mathbb{M})$. We indicate with $\Gamma(f)$ the quadratic differential form defined by (1.4) and denote by $d(x, y)$ the canonical distance associated with L as in (1.3) in the introduction.

There is another useful distance on \mathbb{M} which in fact coincides with $d(x, y)$. Such distance is based on the notion of subunit curve introduced by Fefferman and Phong in [FP], see also [JSC2]. By a result in [PS], given any point $x \in \mathbb{M}$ there exists an open set $x \in U \subset \mathbb{M}$ in which the operator L can be written as

$$(2.1) \quad L = - \sum_{i=1}^m X_i^* X_i,$$

where the vector fields X_i have Lipschitz continuous coefficients in U , and X_i^* indicates the formal adjoint of X_i in $L^2(\mathbb{M}, d\mu)$. We remark that such local representation of L is not unique. A tangent vector $v \in T_x \mathbb{M}$ is called subunit for L at x if $v = \sum_{i=1}^m a_i X_i(x)$, with $\sum_{i=1}^m a_i^2 \leq 1$. It turns out that the notion of subunit vector for L at x does not depend on the local representation (2.1) of L . A Lipschitz path $\gamma : [0, T] \rightarrow \mathbb{M}$ is called subunit for L if $\gamma'(t)$ is subunit for L at $\gamma(t)$ for a.e. $t \in [0, T]$. We then define the subunit length of γ as $\ell_s(\gamma) = T$. Given $x, y \in \mathbb{M}$, we indicate with

$$S(x, y) = \{\gamma : [0, T] \rightarrow \mathbb{M} \mid \gamma \text{ is subunit for } L, \gamma(0) = x, \gamma(T) = y\}.$$

In this paper we assume that $S(x, y) \neq \emptyset$ for every $x, y \in \mathbb{M}$. Under such assumption it is easy to verify that

$$(2.2) \quad d_s(x, y) = \inf\{\ell_s(\gamma) \mid \gamma \in S(x, y)\},$$

defines a true distance on \mathbb{M} . Furthermore, thanks to Lemma 5.43 in [CKS] we know that

$$d(x, y) = d_s(x, y), \quad x, y \in \mathbb{M},$$

hence we can work indifferently with either one of the distances d or d_s .

In addition to the differential form (1.4), we assume that \mathbb{M} be endowed with another smooth bilinear differential form, indicated with Γ^Z , satisfying for $f, g \in C^\infty(\mathbb{M})$

$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h),$$

and $\Gamma^Z(f) = \Gamma^Z(f, f) \geq 0$. We make the following assumptions that will be in force throughout the paper:

(H.1) There exists an increasing sequence $h_k \in C_0^\infty(\mathbb{M})$ such that $h_k \nearrow 1$ on \mathbb{M} , and

$$\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

(H.2) For any $f \in C^\infty(\mathbb{M})$ one has

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

(H.3) The *generalized curvature-dimension inequality* $CD(\rho_1, \rho_2, \kappa, d)$ be satisfied with $\rho_1 \geq 0$, that is: There exist constants $\rho_1 \geq 0$, $\rho_2 > 0$, $\kappa \geq 0$, and $d \geq 2$ such that the inequality

$$\Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \frac{1}{d} (Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu} \right) \Gamma(f) + \rho_2 \Gamma^Z(f)$$

hold for every $f \in C^\infty(\mathbb{M})$ and every $\nu > 0$, where Γ_2 and Γ_2^Z are defined by 1.5 and 1.6.

For example, as we mentioned in the introduction, the assumptions (H.1)-(H.3) are satisfied in all Carnot groups of step two, and in all complete Sasakian manifolds whose horizontal Tanaka-Webster Ricci curvature is non negative. For further examples, including a wide class of bundles over Riemannian manifolds we refer the reader to [BG1].

2.2. Preliminary results. In what follows we collect some results from [BG1] which will be needed in this paper. In the framework of Section 2.1, L is essentially self-adjoint on $C_0^\infty(\mathbb{M})$. Furthermore, the heat semigroup $(P_t)_{t \geq 0}$ with infinitesimal generator L is stochastically complete, i.e. $P_t 1 = 1$ (see [BG1]). Due to the hypoellipticity of L , the function $(t, x) \rightarrow P_t f(x)$ is smooth on $\mathbb{M} \times (0, \infty)$ and

$$P_t f(x) = \int_{\mathbb{M}} p(x, y, t) f(y) d\mu(y), \quad f \in C_0^\infty(\mathbb{M}),$$

where $p(x, y, t) = p(y, x, t) > 0$ is the so-called heat kernel associated to P_t .

Henceforth in this paper we denote

$$C_b^\infty(\mathbb{M}) = C^\infty(\mathbb{M}) \cap L^\infty(\mathbb{M}).$$

For $\varepsilon > 0$ we also denote by \mathcal{A}_ε the set of functions $f \in C_b^\infty(\mathbb{M})$ such that

$$f = g + \varepsilon,$$

for some $\varepsilon > 0$ and some $g \in C_b^\infty(\mathbb{M})$, $g \geq 0$, such that $g, \sqrt{\Gamma(g)}, \sqrt{\Gamma^Z(g)} \in L^2(\mathbb{M})$. As shown in [BG1], this set is stable under the action of P_t , i.e., if $f \in \mathcal{A}_\varepsilon$, then $P_t f \in \mathcal{A}_\varepsilon$.

Let us fix $x \in \mathbb{M}$ and $T > 0$. Given a function $f \in \mathcal{A}_\varepsilon$, for $0 \leq t \leq T$ we introduce the entropy functionals

$$\Phi_1(t) = P_t((P_{T-t} f) \Gamma(\ln P_{T-t} f))(x),$$

$$\Phi_2(t) = P_t((P_{T-t} f) \Gamma^Z(\ln P_{T-t} f))(x).$$

For later use, we observe here that, with the above notations,

$$(2.3) \quad \int_0^T \Phi_1(t) dt = P_T(f \ln f)(x) - P_T f(x) \ln P_T f(x).$$

For the sake of brevity, we will often omit reference to the point $x \in \mathbb{M}$, and write for instance $P_T f$ instead of $P_T f(x)$. This should cause no confusion in the reader.

The main source of the functional inequalities that will be studied in the present work is the following result:

Theorem 2.1. *Let $a, b : [0, T] \rightarrow [0, \infty)$ and $\gamma : [0, T] \rightarrow \mathbb{R}$ be C^1 functions. For $\varepsilon > 0$ and $f \in \mathcal{A}_\varepsilon$, we have*

$$\begin{aligned} & a(T)P_T(f\Gamma(\ln f)) + b(T)P_T(f\Gamma^Z(\ln f)) - a(0)(P_T f)\Gamma(\ln P_T f) - b(0)(P_T f)\Gamma^Z(\ln P_T f) \\ & \geq \int_0^T \left(a' + 2\rho_1 a - 2\kappa \frac{a^2}{b} - 4 \frac{a\gamma}{d} \right) \Phi_1 ds + \int_0^T (b' + 2\rho_2 a) \Phi_2 ds \\ & \quad + \frac{4}{d} \int_0^T a\gamma ds \, LP_T f - \frac{2}{d} \int_0^T a\gamma^2 ds \, P_T f. \end{aligned}$$

Proof. This result is proved in [BG1], but since the main argument is short for the sake of completeness we repeat it here. The idea is to introduce the functional

$$\Psi(t) = a(t)\Phi_1(t) + b(t)\Phi_2(t).$$

A direct computation yields

$$\Psi'(t) = a'(t)\Phi_1(t) + b'(t)\Phi_2(t) + 2a(t)P_t((P_{T-t}f)\Gamma_2(\ln P_{T-t}f)) + 2b(t)P_t((P_{T-t}f)\Gamma_2^Z(\ln P_{T-t}f)).$$

By using the inequality $CD(\rho_1, \rho_2, \kappa, d)$, we then obtain the bound

$$\Psi'(t) \geq \left(a' + 2\rho_1 a - 2\kappa \frac{a^2}{b} \right) \Phi_1 + (b' + 2\rho_2 a) \Phi_2 + \frac{2a}{d} (P_t((P_{T-t}f)(L \ln P_{T-t}f)^2))$$

We now have

$$L \ln P_{T-t}f = \frac{LP_{T-t}f}{P_{T-t}f} - \Gamma(\ln P_{T-t}f),$$

and an application of the inequality $x^2 \geq 2xy - y^2$ gives,

$$(L \ln P_{T-t}f)^2 \geq 2\gamma(t)L \ln P_{T-t}f - \gamma^2(t).$$

Therefore,

$$\Psi'(t) \geq \left(a' + 2\rho_1 a - 2\kappa \frac{a^2}{b} - 4 \frac{a\gamma}{d} \right) \Phi_1 + (b' + 2\rho_2 a) \Phi_2 + \frac{4a\gamma}{d} LP_T f - 2 \frac{a\gamma^2}{d} P_T f.$$

Integrating this inequality from 0 to T yields the desired conclusion. \square

Henceforth in this paper, we let

$$(2.4) \quad D = \left(1 + \frac{3\kappa}{2\rho_2} \right) d.$$

The following scale invariant Harnack inequality for the heat kernel was proved in [BG1].

Proposition 2.2. *Let $p(x, y, t)$ be the heat kernel on \mathbb{M} . For every $x, y, z \in \mathbb{M}$ and every $0 < s < t < \infty$ one has*

$$p(x, y, s) \leq p(x, z, t) \left(\frac{t}{s} \right)^{\frac{D}{2}} \exp \left(\frac{D}{d} \frac{d(y, z)^2}{4(t-s)} \right).$$

A basic consequence of this Harnack inequality is the control of the volume growth of balls centered at a given point.

Proposition 2.3. *For every $x \in \mathbb{M}$ and every $R_0 > 0$ there is a constant $C(d, \kappa, \rho_2) > 0$ such that,*

$$\mu(B(x, R)) \leq \frac{C(d, \kappa, \rho_2)}{R_0^D p(x, x, R_0^2)} R^D, \quad R \geq R_0.$$

Proof. Fix $x \in \mathbb{M}$ and $t > 0$. Applying Proposition 2.2 to $p(x, y, t)$ for every $y \in B(x, \sqrt{t})$ we find

$$p(x, x, t) \leq 2^{\frac{D}{2}} e^{\frac{D}{4d}} p(x, y, 2t) = C(d, \kappa, \rho_2) p(x, y, 2t).$$

Integration over $B(x, \sqrt{t})$ gives

$$p(x, x, t) \mu(B(x, \sqrt{t})) \leq C(d, \kappa, \rho_2) \int_{B(x, \sqrt{t})} p(x, y, 2t) d\mu(y) \leq C(d, \kappa, \rho_2),$$

where we have used $P_t 1 \leq 1$. This gives the on-diagonal upper bound

$$(2.5) \quad p(x, x, t) \leq \frac{C(d, \kappa, \rho_2)}{\mu(B(x, \sqrt{t}))}.$$

Let now $t > \tau > 0$. Again, from the Harnack inequality of Proposition 2.2, we have

$$p(x, x, t) \geq p(x, x, \tau) \left(\frac{\tau}{t} \right)^{\frac{D}{2}}.$$

The inequality (2.5) finally implies the desired conclusion. \square

2.3. Reverse logarithmic Sobolev inequalities. In this section we derive some functional inequalities which will play a fundamental role in the proof of Theorem 3.1 below.

Proposition 2.4. *Let $\varepsilon > 0$ and $f \in \mathcal{A}_\varepsilon$. For $x \in \mathbb{M}$, $t, \tau > 0$, and $C \in \mathbb{R}$, one has*

$$\begin{aligned} & \frac{\tau}{\rho_2} P_t(f\Gamma(\ln f))(x) + \tau^2 P_t(f\Gamma^Z(\ln f))(x) + \frac{1}{\rho_2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) \left[P_t(f \ln f)(x) - P_t f(x) \ln P_t f(x) \right] \\ & \geq \frac{t + \tau}{\rho_2} P_t f(x) \Gamma(\ln P_t f)(x) + (t + \tau)^2 P_t f(x) \Gamma^Z(\ln P_t f)(x) + \frac{4Ct}{\rho_2 d} L P_t f(x) - \frac{2C^2}{d\rho_2} \ln \left(1 + \frac{t}{\tau} \right) P_t f(x). \end{aligned}$$

Proof. Let $T, \tau > 0$ be arbitrarily fixed. We apply Theorem 2.1, in which we choose

$$b(t) = (T + \tau - t)^2, \quad a(t) = \frac{1}{\rho_2} (T + \tau - t), \quad \gamma(t) = \frac{C}{T + \tau - t}, \quad 0 \leq t \leq T.$$

With such choices we obtain

$$(2.6) \quad \begin{cases} a' - 2\kappa \frac{a^2}{b} - 4 \frac{a\gamma}{d} \equiv -\frac{1}{\rho_2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right), \\ b' + 2\rho_2 a \equiv 0, \\ \int_0^T \frac{4a\gamma}{d} = \frac{4CT}{\rho_2 d}, \\ \text{and} \\ - \int_0^T \frac{2a\gamma^2}{d} = -\frac{2C^2}{d\rho_2} \ln \left(1 + \frac{T}{\tau} \right). \end{cases}$$

Keeping (2.3) in mind, we obtain the sought for conclusion with T in place of t . The arbitrariness of $T > 0$ finishes the proof. \square

A first notable consequence of Proposition 2.4 is the following reverse log-Sobolev inequality.

Corollary 2.5. *Let $\varepsilon > 0$ and $f \in \mathcal{A}_\varepsilon$. For $x \in \mathbb{M}$, $t > 0$ one has*

$$t P_t f(x) \Gamma(\ln P_t f)(x) + \rho_2 t^2 P_t f(x) \Gamma^Z(\ln P_t f)(x) \leq \left(1 + \frac{2\kappa}{\rho_2} \right) [P_t(f \ln f)(x) - P_t f(x) \ln P_t f(x)].$$

Proof. We first apply Proposition 2.4 with $C = 0$, and then we let $\tau \rightarrow 0^+$ in the resulting inequality. \square

We may actually improve Corollary 2.5 and obtain the following crucial dimensional reverse log-Sobolev inequality.

Theorem 2.6. *Let $\varepsilon > 0$ and $f \in \mathcal{A}_\varepsilon$, then for every $C \geq 0$ and $\delta > 0$, one has for $x \in \mathbb{M}$, $t > 0$,*

$$(2.7) \quad \begin{aligned} & \frac{t}{\rho_2} P_t f(x) \Gamma(\ln P_t f)(x) + t^2 P_t f(x) \Gamma^Z(\ln P_t f)(x) \\ & \leq \frac{1}{\rho_2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) [P_t(f \ln f)(x) - P_t f(x) \ln P_t f(x)] \\ & \quad - \frac{4C}{\rho_2 d} \frac{t}{1 + \delta} L P_t f(x) + \frac{2C^2}{d \rho_2} \ln \left(1 + \frac{1}{\delta} \right) P_t f(x). \end{aligned}$$

Proof. For $x \in \mathbb{M}$, $t, \tau > 0$, we apply Proposition 2.4 to the function $P_\tau f$ instead of f . Recalling that $P_t(P_\tau f) = P_{t+\tau} f$, we obtain, for all $C \in \mathbb{R}$,

$$(2.8) \quad \begin{aligned} & \frac{\tau}{\rho_2} P_t(P_\tau f \Gamma(\ln P_\tau f))(x) + \tau^2 P_t(P_\tau f \Gamma^Z(\ln P_\tau f))(x) \\ & \quad + \frac{1}{\rho_2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) [P_t(P_\tau f \ln P_\tau f)(x) - P_{t+\tau} f(x) \ln P_{t+\tau} f(x)] \\ & \geq \frac{t + \tau}{\rho_2} P_{t+\tau} f(x) \Gamma(\ln P_{t+\tau} f)(x) + (t + \tau)^2 P_{t+\tau} f(x) \Gamma^Z(\ln P_{t+\tau} f)(x) \\ & \quad + \frac{4C}{\rho_2 d} t L P_{t+\tau} f(x) - \frac{2C^2}{d \rho_2} \ln \left(1 + \frac{t}{\tau} \right) P_{t+\tau} f(x). \end{aligned}$$

Invoking Proposition 2.5 we now find for every $x \in \mathbb{M}$, $\tau > 0$,

$$\tau P_\tau f(x) \Gamma(\ln P_\tau f)(x) + \rho_2 \tau^2 P_\tau f(x) \Gamma^Z(\ln P_\tau f)(x) \leq \left(1 + \frac{2\kappa}{\rho_2} \right) [P_\tau(f \ln f)(x) - P_\tau f(x) \ln P_\tau f(x)].$$

If we now apply P_t to this inequality, we obtain

$$\tau P_t(P_\tau f \Gamma(\ln P_\tau f))(x) + \rho_2 \tau^2 P_t(P_\tau f \Gamma^Z(\ln P_\tau f))(x) \leq \left(1 + \frac{2\kappa}{\rho_2} \right) [P_{t+\tau}(f \ln f)(x) - P_t(P_\tau f \ln P_\tau f)(x)].$$

We use this inequality to bound from above the first two terms in the left-hand side of (2.8), obtaining

$$\begin{aligned} & \frac{1 + \frac{2\kappa}{\rho_2}}{\rho_2} P_{t+\tau}(f \ln f)(x) + \frac{4C}{\rho_2 d} P_t(P_\tau f \ln P_\tau f)(x) - \frac{1}{\rho_2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) P_{t+\tau} f(x) \ln P_{t+\tau} f(x) \\ & \geq \frac{t + \tau}{\rho_2} P_{t+\tau} f(x) \Gamma(\ln P_{t+\tau} f)(x) + (t + \tau)^2 P_{t+\tau} f(x) \Gamma^Z(\ln P_{t+\tau} f)(x) \\ & \quad + \frac{4C}{\rho_2 d} t L P_{t+\tau} f(x) - \frac{2C^2}{d \rho_2} \ln \left(1 + \frac{t}{\tau} \right) P_{t+\tau} f(x). \end{aligned}$$

Consider the convex function $\Phi(s) = s \ln s$, $s > 0$. Thanks to Jensen's inequality, we have for any $\tau > 0$ and $x \in \mathbb{M}$

$$\Phi(P_\tau f(x)) \leq P_\tau(\Phi(f))(x),$$

which we can rewrite

$$P_\tau f(x) \ln P_\tau f(x) \leq P_\tau(f \ln f)(x).$$

For $C \geq 0$, applying P_t to this inequality we find

$$\frac{4C}{\rho_2 d} P_t(P_\tau f \ln P_\tau f)(x) \leq \frac{4C}{\rho_2 d} P_{t+\tau}(f \ln f)(x).$$

We therefore conclude, for $C \geq 0$,

$$\begin{aligned} & \frac{1}{\rho_2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) [P_{t+\tau}(f \ln f)(x) - P_{t+\tau}f(x) \ln P_{t+\tau}f(x)] \\ & \geq \frac{t+\tau}{\rho_2} P_{t+\tau}f(x) \Gamma(\ln P_{t+\tau}f)(x) + (t+\tau)^2 P_{t+\tau}f(x) \Gamma^Z(\ln P_{t+\tau}f)(x) \\ & \quad + \frac{4C}{\rho_2 d} t L P_{t+\tau}f(x) - \frac{2C^2}{d \rho_2} \ln \left(1 + \frac{t}{\tau} \right) P_{t+\tau}f(x). \end{aligned}$$

If in the latter inequality we now choose $\tau = \delta t$, we find:

$$\begin{aligned} & \frac{1}{\rho_2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) [P_{t+\delta t}(f \ln f)(x) - P_{t+\delta t}f(x) \ln P_{t+\delta t}f(x)] \\ & \geq \frac{t+\delta t}{\rho_2} P_{t+\delta t}f(x) \Gamma(\ln P_{t+\delta t}f)(x) + (t+\delta t)^2 P_{t+\delta t}f(x) \Gamma^Z(\ln P_{t+\delta t}f)(x) \\ & \quad + \frac{4C}{\rho_2 d} t L P_{t+\delta t}f(x) - \frac{2C^2}{d \rho_2} \ln \left(1 + \frac{1}{\delta} \right) P_{t+\delta t}f(x). \end{aligned}$$

Changing $(1+\delta)t$ into t in the latter inequality, we finally conclude:

$$\begin{aligned} & \frac{t}{\rho_2} P_t f(x) \Gamma(\ln P_t f)(x) + t^2 P_t f(x) \Gamma^Z(\ln P_t f)(x) \\ & \leq \frac{1}{\rho_2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) [P_t(f \ln f)(x) - P_t f(x) \ln P_t f(x)] \\ & \quad - \frac{4C}{\rho_2 d} \frac{t}{1+\delta} L P_t f(x) + \frac{2C^2}{d \rho_2} \ln \left(1 + \frac{1}{\delta} \right) P_t f(x). \end{aligned}$$

This gives the desired conclusion (2.7). □

3. VOLUME DOUBLING PROPERTY

Our principal objective of this section is proving the following result.

Theorem 3.1 (Global doubling property). *The metric measure space (\mathbb{M}, d, μ) satisfies the global volume doubling property. More precisely, there exists a constant $C_1 = C_1(\rho_1, \rho_2, \kappa, d) > 0$ such that for every $x \in \mathbb{M}$ and every $r > 0$,*

$$\mu(B(x, 2r)) \leq C_1 \mu(B(x, r)).$$

3.1. Small time asymptotics. As a first step, we prove a small time asymptotics result interesting in itself, see Proposition 3.3 below. In what follows for a given set $A \subset \mathbb{M}$ we will denote by $\mathbf{1}_A$ its indicator function. We will need the following simple form of the maximum principle.

Lemma 3.2. *Fix a base point $x \in \mathbb{M}$, and let $d(y) = d(y, x)$. For $\alpha > 0$, let f be a measurable function on \mathbb{M} , $f \geq 0$, such that*

$$(3.1) \quad \int_{\mathbb{M}} e^{2\alpha d(y)} f(y)^2 d\mu(y) < \infty.$$

Then, $f \in L^1(\mathbb{M}) \cap L^2(\mathbb{M})$ and if we let $u(y, t) = P_t f(y)$, then for $t > 0$ we have

$$(3.2) \quad \int_{\mathbb{M}} e^{2\alpha d(y)} u(y, t)^2 d\mu(y) \leq e^{2\alpha^2 t} \int_{\mathbb{M}} e^{2\alpha d(y)} f(y)^2 d\mu(y).$$

If instead $\alpha < 0$ and $f \in L^\infty(\mathbb{M})$, then $u(y, t) = P_t f(y)$ belongs to $L^\infty(\mathbb{M} \times (0, \infty))$, and the same conclusion (3.2) is valid in this case.

Proof. We first consider the case in which $\alpha > 0$, and f satisfies (3.1). Since in such case $e^{2\alpha d} \geq 1$, we easily see that $f \in L^2(\mathbb{M})$. To verify that $f \in L^1(\mathbb{M})$ we argue as follows. From the Cauchy-Schwarz inequality we have

$$\int_{\mathbb{M}} |f(y)| d\mu(y) \leq \left(\int_{\mathbb{M}} e^{2\alpha d(y)} f(y)^2 d\mu(y) \right)^{1/2} \left(\int_{\mathbb{M}} e^{-2\alpha d(y)} d\mu(y) \right)^{1/2}.$$

In view of (3.1) it will thus suffice that the second integral in the right-hand side be finite. Now

$$\begin{aligned} \int_{\mathbb{M}} e^{-2\alpha d(y)} d\mu(y) &= \mu(B(x, 1)) + \sum_{i=0}^{\infty} \int_{B(x, 2^{i+1}) \setminus B(x, 2^i)} e^{-2\alpha d(y)} d\mu(y) \\ &\leq \mu(B(x, 1)) + \sum_{i=0}^{\infty} e^{-(2^{i+1}\alpha)} \mu(B(x, 2^{i+1})). \end{aligned}$$

Proposition 2.3 guarantees the existence of a constant $C(x) > 0$ such that for any $R \geq 1$,

$$\mu(B(x, R)) \leq C(x)R^D.$$

As a consequence, for every $i = 0, 1, 2, \dots$, we obtain

$$\mu(B(x, 2^{i+1})) \leq C(x)(2^{i+1})^D.$$

This gives,

$$\begin{aligned} \int_{\mathbb{M}} e^{-2\alpha d(y)} d\mu(y) &\leq \mu(B(x, 1)) + C(x) \sum_{i=0}^{\infty} (2^{i+1})^D e^{-(2^{i+1}\alpha)} \\ &= \mu(B(x, 1)) + C(x) 2^D e^{-2\alpha} \sum_{i=0}^{\infty} (2^i)^D e^{-((2^i-1)2\alpha)} \\ &\leq \mu(B(x, 1)) + C'(x) e^{-2\alpha} \sum_{k=1}^{\infty} k^D e^{-2\alpha(k-1)} \\ &\leq \mu(B(x, 1)) + C'(x) e^{-2\alpha} \int_0^{\infty} (x+1)^D e^{-2\alpha x} dx \\ &\leq \mu(B(x, 1)) + C'(x) e^{-2\alpha} \sum_{k=0}^{[D]+1} \binom{[D]+1}{k} \int_0^{\infty} x^k e^{-2\alpha x} dx \\ &= \mu(B(x, 1)) + C'(x) e^{-2\alpha} \sum_{k=0}^{[D]+1} \binom{[D]+1}{k} (2\alpha)^{-(k+1)} \Gamma(k+1) < \infty, \end{aligned}$$

where we have denoted by $\Gamma(z)$ the value of Euler's Gamma function at $z > 0$. This proves that $f \in L^1(\mathbb{M})$.

Our next objective is proving (3.2). At first, we prove (3.2) under the additional assumption that f be compactly supported (not necessarily smooth). We pick a sequence of relatively compact open sets Ω_n such that $\bar{\Omega}_n \subset \Omega_{n+1}$ and

$$\bigcup_{n=1}^{\infty} \Omega_n = \mathbb{M}.$$

We fix n large enough so that Ω_n contain the support of f . We introduce the functional

$$J_n(t) = \int_{\Omega_n} e^{2\alpha d(y)} u_n(y, t)^2 d\mu(y) < \infty,$$

where P^{Ω_n} is the Dirichlet semigroup on Ω_n , and $u_n(y, t) = P_t^{\Omega_n} f(y)$. We note that u_n solves the Cauchy-Dirichlet problem

$$(3.3) \quad \begin{cases} Lu_n - (u_n)_t = 0, & \text{in } \Omega_n \times (0, \infty), \\ u_n = 0, & \text{on } \partial\Omega_n \times (0, \infty), \\ u_n(y, 0) = f(y), & y \in \Omega_n. \end{cases}$$

The interpretation of (3.3) in the sense of (weak) Sobolev spaces implies that for every $t \in (0, \infty)$ one has $u_n(\cdot, t) \in \overset{o}{W}_H^{1,2}(\Omega_n)$ and moreover

$$\int_{\Omega_n} [\Gamma(u_n(\cdot, t), \phi)(y) + (u_n)_t(y, t)\phi(y)] d\mu(y) = 0,$$

for every $\phi \in \overset{o}{W}_H^{1,2}(\Omega_n)$. If for $t > 0$ fixed we take $\phi(y) = e^{2\alpha d(y)} u_n(y, t)$, then for $u \in \overset{o}{W}_H^{1,2}(\Omega_n)$,

$$\Gamma(u, \phi) = e^{2\alpha d} \Gamma(u, u_n) + 2\alpha e^{2\alpha d} u_n \Gamma(u, d)$$

We notice here that since the function $y \rightarrow d(y)$ is Lipschitz continuous (with respect to the Carnot-Carathéodory distance), by the Rademacher theorem in [GN2] we conclude that $y \rightarrow e^{2\alpha d(y)}$ belongs to the Sobolev space $\overset{o}{W}_H^{1,\infty}(\Omega_n)$, and therefore by the Leibniz rule the function $y \rightarrow e^{2\alpha d(y)} u_n(y, t)$ belongs to $\overset{o}{W}_H^{1,2}(\Omega_n)$. This gives

$$\begin{aligned} 0 &= \int_{\Omega_n} [\Gamma(u_n(\cdot, t), \phi)(y) + (u_n)_t(y, t)\phi(y)] d\mu(y) \\ &= \int_{\Omega_n} e^{2\alpha d(y)} \left[\Gamma(u_n(\cdot, t))(y) + 2\alpha u_n(y, t) \Gamma(u_n(\cdot, t), d)(y) + \frac{1}{2} (u_n^2(\cdot, t))_t(y) \right] d\mu(y). \end{aligned}$$

Since

$$J'_n(t) = \int_{\Omega_n} e^{2\alpha d(y)} (u_n^2(\cdot, t))_t(y) d\mu(y),$$

we conclude that

$$(3.4) \quad J'_n(t) = -2 \int_{\Omega_n} e^{2\alpha d(y)} [\Gamma(u_n(\cdot, t))(y) + 2\alpha u_n(y, t) \Gamma(u_n(\cdot, t), d)(y)] d\mu(y).$$

In the sequel we will omit reference to the variable of integration $y \in \mathbb{M}$. Applying Cauchy-Schwarz inequality in (3.4), and using the fact that $\Gamma(d) \leq 1$ μ a. e. in \mathbb{M} , we find

$$\begin{aligned} J'_n(t) &\leq -2 \int_{\Omega_n} e^{2\alpha d} \Gamma(u_n) d\mu + 4\alpha \left(\int_{\Omega_n} e^{2\alpha d} \Gamma(u_n) d\mu \right)^{1/2} \left(\int_{\Omega_n} e^{2\alpha d} u_n^2 \Gamma(d) d\mu \right)^{1/2} \\ &\leq 2(\alpha\varepsilon - 1) \int_{\Omega_n} e^{2\alpha d} \Gamma(u_n) d\mu + \frac{2\alpha}{\varepsilon} J_n(t), \end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Choosing $\varepsilon = \alpha^{-1}$, we obtain the inequality

$$J'_n(t) \leq 2\alpha^2 J_n(t).$$

Integrating such inequality on the interval $[0, t]$, where $t > 0$ is arbitrarily fixed, we reach the conclusion

$$(3.5) \quad J_n(t) \leq e^{2\alpha^2 t} J_n(0) = e^{2\alpha^2 t} \int_{\Omega_n} e^{2\alpha d(y)} f^2(y) d\mu(y) \leq e^{2\alpha^2 t} \int_{\mathbb{M}} e^{2\alpha d(y)} f^2(y) d\mu(y).$$

Since by the weak maximum principle, we have for $n \leq q$

$$P_t^{\Omega_n} f(y) \leq P_t^{\Omega_q} f(y), \quad (y, t) \in \Omega_n \times (0, \infty),$$

the sequence $(P_t^{\Omega_n} f)^2$ is increasing and, as $n \rightarrow \infty$, it converges pointwise μ a.e. to $(P_t f)^2$ and therefore this is also the same for $\mathbf{1}_{\Omega_n}(P_t^{\Omega_n} f)^2$. Letting $n \rightarrow \infty$ in the latter inequality, by the monotone convergence theorem, we thus obtain

$$\int_{\mathbb{M}} e^{2\alpha d(y)} P_t f(y)^2 d\mu(y) \leq e^{2\alpha^2 t} \int_{\mathbb{M}} e^{2\alpha d(y)} f(y)^2 d\mu(y).$$

This proves (3.2) when f is compactly supported. If now f is a measurable function of \mathbb{M} satisfying (3.1), we apply the inequality (3.2) to $h_n f$, where $0 \leq h_n \leq 1$, h_n compactly supported and $h_n \nearrow 1$. This gives

$$\int_{\mathbb{M}} e^{2\alpha d(y)} P_t(h_n f)(y)^2 d\mu(y) \leq e^{2\alpha^2 t} \int_{\mathbb{M}} e^{2\alpha d(y)} h_n^2(y) f(y)^2 d\mu(y).$$

Since $h_n^2 \nearrow 1$, we reach the desired conclusion for the function f by the monotone convergence theorem.

Finally, when $\alpha < 0$, then the function $e^{2\alpha d} \in L^1(\mathbb{M})$ (see the proof above), and using the fact that, if $f \in L^\infty(\mathbb{M})$, then $u \in L^\infty(\mathbb{M} \times (0, \infty))$, the same computations as above easily give the desired conclusion (3.2) in this case. \square

We now turn to the announced small time asymptotics.

Proposition 3.3. *Given $x \in \mathbb{M}$ and $r > 0$, let $f = \mathbf{1}_{B(x, r)^c}$. One has,*

$$\liminf_{s \rightarrow 0^+} (-s \ln P_s f(x)) \geq \frac{r^2}{4}.$$

Proof. To prove the proposition it will suffice to show that

$$\limsup_{t \rightarrow 0^+} (t \ln P_t f(x)) \leq -\frac{r^2}{4}.$$

Let $\alpha > 0$ to be suitably chosen subsequently. From the group property of P_t and the Cauchy-Schwarz inequality we find

$$\begin{aligned} (3.6) \quad P_t \mathbf{1}_{B(x, r)^c}(x)^2 &= P_{\frac{t}{2}}(P_{\frac{t}{2}} \mathbf{1}_{B(x, r)^c})(x)^2 \\ &= \left(\int_{\mathbb{M}} p(x, y, t/2) P_{\frac{t}{2}} \mathbf{1}_{B(x, r)^c}(y) d\mu(y) \right)^2 \\ &\leq \int_{\mathbb{M}} p(x, y, t/2)^2 e^{2\alpha d(y)} d\mu(y) \int_{\mathbb{M}} e^{-2\alpha d(y)} P_{t/2} \mathbf{1}_{B(x, r)^c}(y)^2 d\mu(y). \end{aligned}$$

We first bound the term $\int_{\mathbb{M}} p(x, y, t/2)^2 e^{2\alpha d(y)} d\mu(y)$ by using the Harnack inequality in Proposition 2.2. Appealing to such result if $\varepsilon > 0$ and $z \in B(x, \varepsilon)$, we find

$$p(x, y, t/2) = p(y, x, t/2) \leq p(y, z, (1 + \varepsilon)t/2) (1 + \varepsilon)^{\frac{D}{2}} \exp\left(\frac{D}{d} \frac{\varepsilon}{2t}\right).$$

Integrating the latter inequality with respect to $z \in B(x, \varepsilon)$, we obtain

$$\mu(B(x, \varepsilon)) p(x, y, t/2) \leq (1 + \varepsilon)^{\frac{D}{2}} \exp\left(\frac{D}{d} \frac{\varepsilon}{2t}\right) \int_{B(x, \varepsilon)} p(y, z, (1 + \varepsilon)t/2) d\mu(z).$$

We have thus proved that for every $\epsilon > 0$, any $x, y \in \mathbb{M}$, and $t > 0$, one has

$$(3.7) \quad p(x, y, t/2) \leq \frac{(1 + \varepsilon)^{\frac{D}{2}} \exp\left(\frac{D}{d} \frac{\varepsilon}{2t}\right)}{\mu(B(x, \varepsilon))} P_{\frac{(1+\varepsilon)t}{2}} (\mathbf{1}_{B(x, \varepsilon)})(y).$$

From (3.7), and an application of (3.2) in Lemma 3.2 to the function $f(y) = \mathbf{1}_{B(x,\varepsilon)}(y)$, we obtain:

$$\begin{aligned}
 (3.8) \quad \int_{\mathbb{M}} p(x, y, t/2)^2 e^{2\alpha d(y)} \mu(dy) &\leq \frac{(1+\varepsilon)^D \exp\left(\frac{D\varepsilon}{d} \frac{t}{2}\right)}{\mu(B(x, \varepsilon))^2} \int_{\mathbb{M}} P_{(1+\varepsilon)t/2}(\mathbf{1}_{B(x, \varepsilon)})(y)^2 e^{2\alpha d(y)} d\mu(y) \\
 &\leq \frac{(1+\varepsilon)^D \exp\left(\frac{D\varepsilon}{d} \frac{t}{2}\right)}{\mu(B(x, \varepsilon))^2} e^{(1+\varepsilon)\alpha^2 t} \int_{B(x, \varepsilon)} e^{2\alpha d(y)} d\mu(y) \\
 &\leq \frac{(1+\varepsilon)^D \exp\left(\frac{D\varepsilon}{d} \frac{t}{2}\right)}{\mu(B(x, \varepsilon))} e^{(1+\varepsilon)\alpha^2 t} e^{2\alpha\varepsilon}.
 \end{aligned}$$

On the other hand, applying the second part of Lemma 3.2 to the function $f(y) = \mathbf{1}_{B(x, r)^c}(y)$, we find

$$\begin{aligned}
 (3.9) \quad \int_{\mathbb{M}} e^{-2\alpha d(y)} P_{t/2} \mathbf{1}_{B(x, r)^c}(y)^2 d\mu(y) &\leq e^{\alpha^2 t} \int_{B(x, r)^c} e^{-2\alpha d(y)} d\mu(y) \\
 &\leq e^{\alpha^2 t} \sum_{i=0}^{\infty} \int_{B(x, 2^{i+1}r) \setminus B(x, 2^i r)} e^{-2\alpha d(y)} d\mu(y) \\
 &\leq e^{\alpha^2 t} \sum_{i=0}^{\infty} \mu(B(x, 2^{i+1}r)) e^{-2^{i+1}\alpha r}.
 \end{aligned}$$

If we now use (3.8) and (3.9) in the estimate (3.6), we obtain

$$(3.10) \quad P_t \mathbf{1}_{B(x, r)^c}(x)^2 \leq \frac{(1+\varepsilon)^D \exp\left(\frac{D\varepsilon}{d} \frac{t}{2}\right)}{\mu(B(x, \varepsilon))} e^{(2+\varepsilon)\alpha^2 t} e^{2\alpha\varepsilon} \sum_{i=0}^{\infty} \mu(B(x, 2^{i+1}r)) e^{-2^{i+1}\alpha r}.$$

At this point, by an estimate similar to that in the proof of Lemma 3.2, we find

$$\sum_{i=0}^{\infty} \mu(B(x, 2^{i+1}r)) e^{-2^{i+1}\alpha r} \leq C'(x, r) e^{-2\alpha r} \sum_{k=0}^{[D]+1} \binom{[D]+1}{k} (2\alpha r)^{-(k+1)} \Gamma(k+1).$$

Substituting this inequality in (3.10) we conclude

$$\begin{aligned}
 (3.11) \quad P_t \mathbf{1}_{B(x, r)^c}(x)^2 &\leq C'(x, r) \frac{(1+\varepsilon)^D \exp\left(\frac{D\varepsilon}{d} \frac{t}{2}\right)}{\mu(B(x, \varepsilon))} e^{2\alpha\varepsilon} \\
 &\quad \times e^{(2+\varepsilon)\alpha^2 t} e^{-2\alpha r} \sum_{k=0}^{[D]+1} \binom{[D]+1}{k} (2\alpha r)^{-(k+1)} \Gamma(k+1).
 \end{aligned}$$

If in (3.11) we choose $\alpha = r/2t$, we finally obtain

$$\limsup_{t \rightarrow 0^+} (t \ln P_t \mathbf{1}_{B(x, r)^c}(x)) \leq \frac{D}{2d} \varepsilon + \varepsilon \frac{r^2}{8} - \frac{r^2}{4}.$$

We conclude by letting $\varepsilon \rightarrow 0$. \square

3.2. Reverse Harnack inequalities. As a second step toward the proof of Theorem 3.1 we investigate some of the consequences of the reverse log-Sobolev inequality in Proposition 2.6 for functions f such that $0 \leq f \leq 1$ (later, we will apply this to indicator functions).

Proposition 3.4. *Let $\varepsilon > 0$, $f \in \mathcal{A}_\varepsilon$, $0 \leq f \leq 1$, and consider the function $u(x, t) = \sqrt{-\ln P_t f(x)}$. Then,*

$$2tu_t + u + \left(1 + \sqrt{\frac{D^*}{2}}\right) u^{1/3} + \sqrt{\frac{D^*}{2}} u^{-1/3} \geq 0,$$

where

$$D^* = d \left(1 + \frac{2\kappa}{\rho_2} \right).$$

Proof. Noting that we have

$$\frac{t}{\rho_2} P_t f(x) \Gamma(\ln P_t f)(x) + t^2 P_t f(x) \Gamma^Z(\ln P_t f)(x) \geq 0,$$

applying the inequality (2.7) in Theorem 2.6, we obtain that for all $C \geq 0$,

$$\frac{d}{2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) P_t(f \ln f)(x) - \frac{d}{2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) (P_t f) \ln P_t f - \frac{2Ct}{1+\delta} LP_t f + \frac{C^2}{\delta} P_t f \geq 0,$$

where we used the fact that

$$\ln \left(1 + \frac{1}{\delta} \right) \leq \frac{1}{\delta}.$$

On the other hand, the hypothesis $0 \leq f \leq 1$ implies $f \ln f \leq 0$. After dividing both sides of the above inequality by $P_t f$, we thus find

$$-\frac{d}{2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} \right) \ln P_t f - \frac{2Ct}{1+\delta} \frac{LP_t f}{P_t f} + \frac{C^2}{\delta} \geq 0.$$

Dividing both sides by $C > 0$, this may be re-written

$$(3.12) \quad -\frac{D^*}{2C} \ln P_t f - 2 \ln P_t f - \frac{2t}{1+\delta} \frac{LP_t f}{P_t f} + \frac{C}{\delta} \geq 0.$$

We now minimize the left-hand side of (3.12) with respect to C . The minimum value is attained in

$$C = \sqrt{-\frac{\delta D^*}{2} \ln P_t f}.$$

Substituting this value in (3.12), we obtain

$$\sqrt{\frac{2D^*}{\delta}} \sqrt{-\ln P_t f} - 2 \ln P_t f - \frac{2t}{1+\delta} \frac{LP_t f}{P_t f} \geq 0.$$

With $u(x, t) = \sqrt{-\ln P_t f(x)}$, and noting that $u_t = -\frac{1}{2u} \frac{LP_t f}{P_t f}$, we can re-write this inequality as follows,

$$\sqrt{\frac{D^*}{2\delta}} + u + \frac{2t}{1+\delta} u_t \geq 0,$$

or equivalently,

$$2tu_t + u + \delta u + (1+\delta) \sqrt{\frac{D^*}{2\delta}} \geq 0.$$

Finally, if we choose

$$\delta = \frac{1}{u^{2/3}},$$

we obtain the desired conclusion. \square

We now introduce the function $g : (0, \infty) \rightarrow (0, \infty)$ defined by

$$g(v) = \frac{1}{v + \left(1 + \sqrt{\frac{D^*}{2}} \right) v^{1/3} + \sqrt{\frac{D^*}{2}} v^{-1/3}}.$$

One easily verifies that

$$\lim_{v \rightarrow 0^+} \sqrt{\frac{D^*}{2}} v^{-1/3} g(v) = 1, \quad \lim_{v \rightarrow \infty} v g(v) = 1.$$

These limit relations show that $g \in L^1(0, A)$ for every $A > 0$, but $g \notin L^1(0, \infty)$. Moreover, if we set

$$G(u) = \int_0^u g(v)dv,$$

then $G'(u) = g(u) > 0$, and thus $G : (0, \infty) \rightarrow (0, \infty)$ is invertible. Furthermore, as $u \rightarrow \infty$ we have

$$(3.13) \quad G(u) = \ln u + C_0 + R(u),$$

where C_0 is a constant and $\lim_{u \rightarrow \infty} R(u) = 0$. At this point we notice that, in terms of the function $g(u)$, we can re-express the conclusion of Proposition 3.4 in the form

$$2tu_t + \frac{1}{g(u)} \geq 0.$$

Keeping in mind that $g(u) = G'(u)$, we thus conclude

$$(3.14) \quad \frac{dG(u)}{dt} = G'(u)u_t \geq -\frac{1}{2t}.$$

From this identity we now obtain the following basic result.

Corollary 3.5. *Let $f \in L^\infty(\mathbb{M})$, $0 \leq f \leq 1$, then for any $x \in \mathbb{M}$ and $0 < s < t$,*

$$G\left(\sqrt{-\ln P_t f(x)}\right) \geq G\left(\sqrt{-\ln P_s f(x)}\right) - \frac{1}{2} \ln\left(\frac{t}{s}\right).$$

Proof. If $f \in \mathcal{A}_\varepsilon$ for some ε , the inequality is a straightforward consequence of the above results. In fact, keeping in mind that $u(x, t) = \sqrt{-\ln P_t f(x)}$, in order to reach the desired conclusion all we need to do is integrating (3.14) between s and t . Consider now $f \in L^\infty(\mathbb{M})$, $0 \leq f \leq 1$. Let $h_n \in C_0^\infty(\mathbb{M})$, with $0 \leq h_n \leq 1$, and $h_n \nearrow 1$. For $n \geq 0$, $\tau > 0$ and $\varepsilon > 0$, the function

$$(1 - \varepsilon)P_\tau(h_n f) + \varepsilon \in \mathcal{A}_\varepsilon.$$

Therefore,

$$G\left(\sqrt{-\ln P_t((1 - \varepsilon)P_\tau(h_n f) + \varepsilon)(x)}\right) \geq G\left(\sqrt{-\ln P_s((1 - \varepsilon)P_\tau(h_n f) + \varepsilon)(x)}\right) - \frac{1}{2} \ln\left(\frac{t}{s}\right).$$

Letting $\varepsilon \rightarrow 0$, $\tau \rightarrow 0$ and finally $n \rightarrow \infty$, we obtain the desired conclusion for f . This completes the proof. \square

Combining Corollary 3.5 with Proposition 3.3 we obtain the following key estimate.

Proposition 3.6. *Let $x \in \mathbb{M}$ and $r > 0$ be arbitrarily fixed. There exists $C_0^* \in \mathbb{R}$, independent of x and r , such that for any $t > 0$,*

$$G\left(\sqrt{-\ln P_t \mathbf{1}_{B(x,r)^c}(x)}\right) \geq \ln \frac{r}{\sqrt{t}} + C_0^*.$$

Proof. Re-write the inequality claimed in Corollary 3.5 as follows

$$G\left(\sqrt{-\ln P_t f(x)}\right) \geq G\left(\sqrt{-\ln P_s f(x)}\right) + \ln \sqrt{s} - \ln \sqrt{t},$$

where we have presently let $f(y) = \mathbf{1}_{B(x,r)^c}(y)$. Since for this function we have $\lim_{s \rightarrow 0^+} (-\ln P_s f(x)) = \infty$, using (3.13) we see that, for $s \rightarrow 0^+$, the latter inequality is equivalent to

$$G\left(\sqrt{-\ln P_t f(x)}\right) \geq \ln \sqrt{-s \ln P_s f(x)} - \ln \sqrt{t} + C_0 + R(\sqrt{-\ln P_s f(x)}).$$

We now take the \liminf as $s \rightarrow 0^+$ of both sides of this inequality. Applying Proposition 3.3 we deduce

$$G\left(\sqrt{-\ln P_t f(x)}\right) \geq \ln \frac{r}{2} - \ln \sqrt{t} + C_0 = \ln \frac{r}{\sqrt{t}} + C_0^*,$$

where we have let $C_0^* = C_0 - \ln 2$. This establishes the desired conclusion. \square

We are now in a position to prove the central result in this paper.

Theorem 3.7. *There exists a constant $A > 0$ such that for every $x \in \mathbb{M}$, and $r > 0$,*

$$P_{Ar^2}(\mathbf{1}_{B(x,r)})(x) \geq \frac{1}{2}.$$

Proof. By the stochastic completeness of \mathbb{M} we know that $P_1 = 1$. Therefore,

$$P_{Ar^2}(\mathbf{1}_{B(x,r)})(x) = 1 - P_{Ar^2}(\mathbf{1}_{B(x,r)^c})(x).$$

We conclude that the desired estimate is equivalent to proving that there exists an absolute constant $A > 0$ such that

$$\sqrt{\ln 2} \leq \sqrt{-\ln P_{Ar^2}(\mathbf{1}_{B(x,r)^c})(x)},$$

or, equivalently,

$$(3.15) \quad G\left(\sqrt{\ln 2}\right) \leq G\left(\sqrt{-\ln P_{Ar^2}(\mathbf{1}_{B(x,r)^c})(x)}\right).$$

At this point we invoke Proposition 3.6, which gives

$$G\left(\sqrt{-\ln P_{Ar^2}(\mathbf{1}_{B(x,r)^c})(x)}\right) \geq \ln\left(\frac{1}{\sqrt{A}} + C_1\right).$$

It is thus clear that, letting $A \rightarrow 0^+$, we can certainly achieve (3.15), thus completing the proof. \square

With Theorem 3.7 in hands we can finally prove Theorem 3.1.

Proof of Theorem 3.1. The argument which shows how to obtain Theorem 3.1 from Theorem 3.7 was developed independently by Grigor'yan [Gri1] and by Saloff-Coste [SC], and it is by now well-known. However, since it is short for the sake of completeness in what follows we provide the relevant details.

From the semigroup property and the symmetry of the heat kernel we have for any $y \in \mathbb{M}$ and $t > 0$

$$p(y, y, 2t) = \int_{\mathbb{M}} p(y, z, t)^2 d\mu(z).$$

Consider now a function $h \in C_0^\infty(\mathbb{M})$ such that $0 \leq h \leq 1$, $h \equiv 1$ on $B(x, \sqrt{t}/2)$ and $h \equiv 0$ outside $B(x, \sqrt{t})$. We thus have

$$\begin{aligned} P_t h(y) &= \int_{\mathbb{M}} p(y, z, t) h(z) d\mu(z) \leq \left(\int_{B(x, \sqrt{t})} p(y, z, t)^2 d\mu(z) \right)^{\frac{1}{2}} \left(\int_{\mathbb{M}} h(z)^2 d\mu(z) \right)^{\frac{1}{2}} \\ &\leq p(y, y, 2t)^{\frac{1}{2}} \mu(B(x, \sqrt{t}))^{\frac{1}{2}}. \end{aligned}$$

If we take $y = x$, and $t = r^2$, we obtain

$$(3.16) \quad P_{r^2}(\mathbf{1}_{B(x,r)})(x)^2 \leq P_{r^2}h(x)^2 \leq p(x, x, 2r^2) \mu(B(x, r)).$$

At this point we use Theorem 3.7, which gives for some $0 < A < 1$, (the fact that we can choose $A < 1$ is clear from the proof of Theorem 3.7)

$$P_{Ar^2}(\mathbf{1}_{B(x,r)})(x) \geq \frac{1}{2}, \quad x \in \mathbb{M}, r > 0.$$

Combining this estimate with the Harnack inequality in Proposition 2.2 and with (3.16), we obtain the following on-diagonal lower bound

$$(3.17) \quad p(x, x, 2r^2) \geq \frac{C^*}{\mu(B(x, r))}, \quad x \in \mathbb{M}, r > 0.$$

Applying Proposition 2.2 we find for every $y \in B(x, \sqrt{t})$,

$$p(x, x, t) \leq C p(x, y, 2t).$$

Integration over $B(x, \sqrt{t})$ gives

$$p(x, x, t) \mu(B(x, \sqrt{t})) \leq C \int_{B(x, \sqrt{t})} p(x, y, 2t) d\mu(y) \leq C,$$

where we have used $P_t 1 \leq 1$. Letting $t = r^2$, we obtain from this the on-diagonal upper bound

$$(3.18) \quad p(x, x, r^2) \leq \frac{C}{\mu(B(x, r))}.$$

Combining (3.17) with (3.18) we finally obtain

$$\mu(B(x, 2r)) \leq \frac{C}{p(x, x, 4r^2)} \leq \frac{CC'}{p(x, x, 2r^2)} \leq C^{**} \mu(B(x, r)),$$

where we have used once more Proposition 2.2, which gives

$$\frac{p(x, x, 2r^2)}{p(x, x, 4r^2)} \leq C',$$

and we have let $C^{**} = CC'(C^*)^{-1}$. This completes the proof. \square

It is well-known that Theorem 3.1 provides the following uniformity control at all scales.

Theorem 3.8. *With C_1 being the constant in Theorem 3.1, let $Q = \log_2 C_1$. For any $x \in \mathbb{M}$ and $r > 0$ one has*

$$\mu(B(x, tr)) \geq C_1^{-1} t^Q \mu(B(x, r)), \quad 0 \leq t \leq 1.$$

4. TWO-SIDED GAUSSIAN BOUNDS, POINCARÉ INEQUALITY AND PARABOLIC HARNACK INEQUALITY

The purpose of this section is to establish some optimal two-sided bounds for the heat kernel $p(x, y, t)$ associated with the subelliptic operator L . Such estimates are reminiscent of those obtained by Li and Yau for complete Riemannian manifolds having $\text{Ric} \geq 0$. As a consequence of the two-sided Gaussian bound for the heat kernel, we will derive the Poincaré inequality and the local parabolic Harnack inequality thanks to well-known results in the works [FS], [KS1], [Gri1], [SC], [St1], [St2], [St3].

We assume, once again, that the assumptions of Section 2.1 are satisfied. Here is our main result.

Theorem 4.1. *For any $0 < \varepsilon < 1$ there exists a constant $C(\varepsilon) = C(d, \kappa, \rho_2, \varepsilon) > 0$, which tends to ∞ as $\varepsilon \rightarrow 0^+$, such that for every $x, y \in \mathbb{M}$ and $t > 0$ one has*

$$\frac{C(\varepsilon)^{-1}}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{Dd(x, y)^2}{d(4-\varepsilon)t}\right) \leq p(x, y, t) \leq \frac{C(\varepsilon)}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{(4+\varepsilon)t}\right).$$

Proof. We begin by establishing the lower bound. First, from Proposition 2.2 we obtain for all $y \in \mathbb{M}$, $t > 0$, and every $0 < \varepsilon < 1$,

$$p(x, y, t) \geq p(x, x, \varepsilon t) \varepsilon^{\frac{D}{2}} \exp\left(-\frac{D}{d} \frac{d(x, y)^2}{(4-\varepsilon)t}\right).$$

We thus need to estimate $p(x, x, \varepsilon t)$ from below. But this has already been done in (3.17). Choosing $r > 0$ such that $2r^2 = \varepsilon t$, we obtain from that estimate

$$p(x, x, \varepsilon t) \geq \frac{C^*}{\mu(B(x, \sqrt{\varepsilon/2}\sqrt{t}))}, \quad x \in \mathbb{M}, t > 0.$$

On the other hand, since $\sqrt{\varepsilon/2} < 1$, by the trivial inequality $\mu(B(x, \sqrt{\varepsilon/2}\sqrt{t})) \leq \mu(B(x, \sqrt{t}))$, we conclude

$$p(x, y, t) \geq \frac{C^*}{\mu(B(x, \sqrt{t}))} \varepsilon^{\frac{D}{2}} \exp\left(-\frac{D}{d} \frac{d(x, y)^2}{(4 - \varepsilon)t}\right).$$

This proves the Gaussian lower bound.

For the Gaussian upper bound, we first observe that the following upper bound is proved in [BG1]:

$$p(x, y, t) \leq \frac{C(d, \kappa, \rho_2, \varepsilon')}{\mu(B(x, \sqrt{t}))^{\frac{1}{2}} \mu(B(y, \sqrt{t}))^{\frac{1}{2}}} \exp\left(-\frac{d(x, y)^2}{(4 + \varepsilon')t}\right).$$

At this point, by the triangle inequality and Theorem 3.8 we find.

$$\begin{aligned} \mu(B(x, \sqrt{t})) &\leq \mu(B(y, d(x, y) + \sqrt{t})) \\ &\leq C_1 \mu(B(y, \sqrt{t})) \left(\frac{d(x, y) + \sqrt{t}}{\sqrt{t}}\right)^Q. \end{aligned}$$

This gives

$$\frac{1}{\mu(B(y, \sqrt{t}))} \leq \frac{C_1}{\mu(B(x, \sqrt{t}))} \left(\frac{d(x, y)}{\sqrt{t}} + 1\right)^Q.$$

Combining this with the above estimate we obtain

$$p(x, y, t) \leq \frac{C_1^{1/2} C(d, \kappa, \rho_2, \varepsilon')}{\mu(B(x, \sqrt{t}))} \left(\frac{d(x, y)}{\sqrt{t}} + 1\right)^{\frac{Q}{2}} \exp\left(-\frac{d(x, y)^2}{(4 + \varepsilon')t}\right).$$

If now $0 < \varepsilon < 1$, it is clear that we can choose $0 < \varepsilon' < \varepsilon$ such that

$$\frac{C_1^{1/2} C(d, \kappa, \rho_2, \varepsilon')}{\mu(B(x, \sqrt{t}))} \left(\frac{d(x, y)}{\sqrt{t}} + 1\right)^{\frac{Q}{2}} \exp\left(-\frac{d(x, y)^2}{(4 + \varepsilon')t}\right) \leq \frac{C^*(d, \kappa, \rho_2, \varepsilon)}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{(4 + \varepsilon)t}\right),$$

where $C^*(d, \kappa, \rho_2, \varepsilon)$ is a constant which tends to ∞ as $\varepsilon \rightarrow 0^+$. The desired conclusion follows by suitably adjusting the values of both ε' and of the constant in the right-hand side of the estimate. \square

With Theorems 3.1 and 4.1 in hands, we can now appeal to the results in [FS], [KS1], [Gri1], [SC], [St1], [St2], [St3], see also the books [GSC], [Gri2]. More precisely, from the developments in these papers it is by now well-known that in the context of strictly regular local Dirichlet spaces we have the equivalence between:

- (1) A two sided Gaussian bounds for the heat kernel (like in Theorem 4.1);
- (2) The conjunction of the volume doubling property and the Poincaré inequality (see Theorem 4.2);
- (3) The parabolic Harnack inequality (see Theorem 4.4).

For uniformly parabolic equations in divergence form the equivalence between (1) and (3) was first proved in [FS]. The fact that (1) implies the volume doubling property is almost straightforward, the argument may be found in [SC2] p. 161. The fact that (1) also implies the Poincaré inequality relies on a beautiful and general argument by Kusuoka and Stroock [KS1], pp. 434-435. The equivalence between (2) and (3) originates from [Gri1] and [SC] and has been worked out in the context of strictly local regular Dirichlet spaces in [St3]. Finally, the fact that (2) implies (1) is also proven in [St3].

Thus, in our framework, thanks to Theorem 4.1 we obtain the following weaker form of Poincaré inequality. Of course we already know the volume doubling property since we proved it to obtain the Gaussian estimates.

Theorem 4.2. *There exists a constant $C = C(d, \kappa, \rho_2) > 0$ such that for every $x \in \mathbb{M}, r > 0$, and $f \in C^\infty(\mathbb{M})$ one has*

$$\int_{B(x,r)} |f(y) - f_r|^2 d\mu(y) \leq Cr^2 \int_{B(x,2r)} \Gamma(f)(y) d\mu(y),$$

where we have let $f_r = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu$.

Since thanks to Theorem 3.1 the space (\mathbb{M}, μ, d) , where $d = d(x, y)$ indicates the sub-Riemannian distance, is a space of homogeneous type, arguing as in [J] we now conclude with the following result.

Corollary 4.3. *There exists a constant $C^* = C^*(d, \kappa, \rho_2) > 0$ such that for every $x \in \mathbb{M}, r > 0$, and $f \in C^\infty(\mathbb{M})$ one has*

$$\int_{B(x,r)} |f(y) - f_r|^2 d\mu(y) \leq C^* r^2 \int_{B(x,r)} \Gamma(f)(y) d\mu(y).$$

Furthermore, the following scale invariant Harnack inequality for local solutions holds.

Theorem 4.4. *If u is a positive solution of the heat equation in a cylinder of the form $Q = (s, s + \alpha r^2) \times B(x, r)$ then*

$$(4.1) \quad \sup_{Q-} u \leq C \inf_{Q+} u,$$

where for some fixed $0 < \beta < \gamma < \delta < \alpha < \infty$ and $\eta \in (0, 1)$,

$$Q- = (s + \beta r^2, s + \gamma r^2) \times B(x, \eta r), Q+ = (s + \delta r^2, s + \alpha r^2) \times B(x, \eta r).$$

Here, the constant C is independent of x, r and u , but depends on the parameters d, κ, ρ_2 , as well as on $\alpha, \beta, \gamma, \delta$ and η .

5. L -HARMONIC FUNCTIONS WITH POLYNOMIAL GROWTH

In this section, we assume again that the assumptions of Section 2.1 are satisfied.

In [BG1] the first and third named authors were able to establish a Yau type Liouville theorem stating that when \mathbb{M} is complete, and the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ holds for $\rho_1 \geq 0$, then there exist no bounded solutions of $Lf = 0$ on \mathbb{M} besides the constants. Note that this result is weaker than Yau's original Riemannian result in [Y1] since this author only assumes a one-side bound. However, as a consequence of Theorem 4.4 we can now remove such limitation and obtain the following complete sub-Riemannian analogue of Yau's Liouville theorem.

Theorem 5.1. *There exist no positive solutions of $Lf = 0$ on \mathbb{M} besides the constants.*

In fact, we can now prove much more. In their celebrated work [CM] Colding and Minicozzi obtained a complete resolution of Yau's famous conjecture that the space of harmonic functions with a fixed polynomial growth at infinity on an open manifold with $\text{Ric} \geq 0$ is finite dimensional. A fundamental discovery in that paper is the fact that such property can be solely derived from the volume doubling condition and the Neumann-Poincaré inequality. In Theorem 8.1 in [CM] the authors, assuming these two properties, present a generalization of their result to sub-Riemannian manifolds. However, since at the time [CM] was written the only application of such theorem that could be given was to Lie groups with polynomial volume growth, see Corollary 8.2 in that paper.

If we combine Theorem 3.1 and Corollary 4.3 in the present work with the cited Theorem 8.1 in [CM], we can considerably broaden the scope of Colding and Minicozzi's result and obtain a positive answer to the following generalization of Yau's conjecture. Given a fixed base point $x_0 \in \mathbb{M}$, and a number $N \in \mathbb{N}$, we will indicate with $\mathcal{H}_N(\mathbb{M}, L)$ the linear space of all solutions of $Lf = 0$ on \mathbb{M} such that there exist a constant $C < \infty$ for which

$$|f(x)| \leq C(1 + d(x, x_0))^N, \quad x \in \mathbb{M}.$$

Theorem 5.2. *For every $N \in \mathbb{N}$ one has*

$$\dim \mathcal{H}_N(\mathbb{M}, L) < \infty.$$

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